Chapter 2

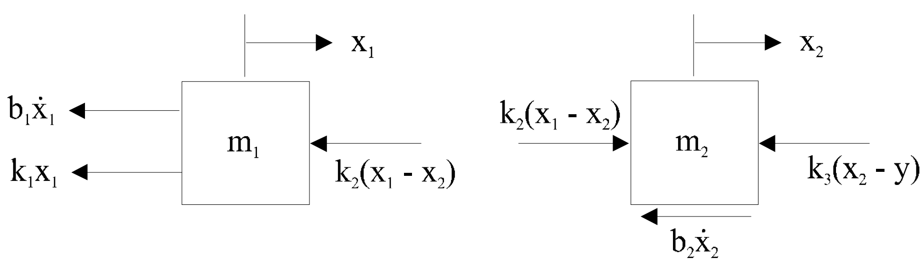
Dynamic Models

Problems and Solutions for Section 2.1

1. Write the differential equations for the mechanical systems shown in Fig. 2.38.

   Solution:

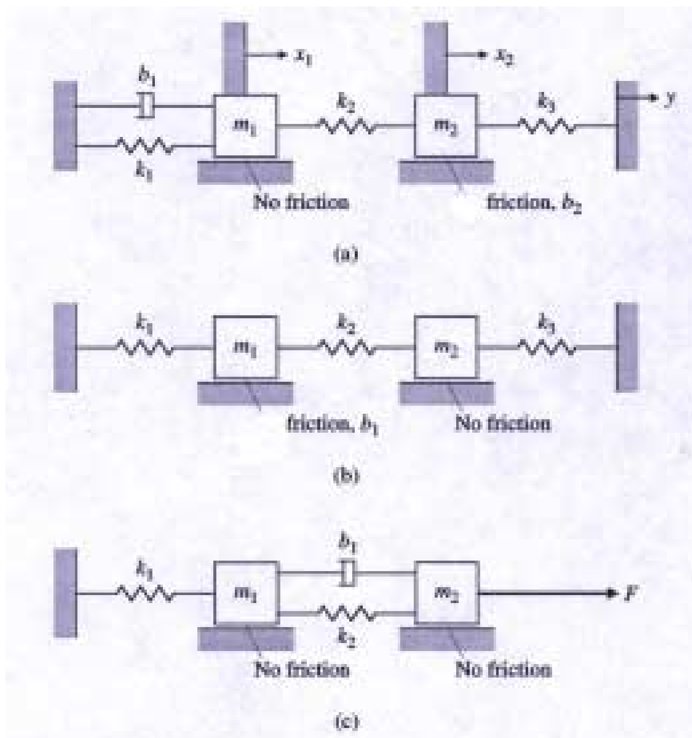
   The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. For (a), to identify the direction of the spring forces on the object, let $x_2 = 0$ and fixed and increase x_1 from 0. Then the k_1 spring will be stretched producing its spring force to the left and the k_2 spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.

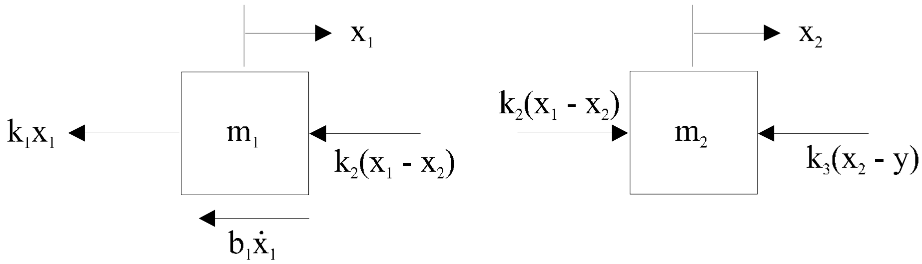


(a)

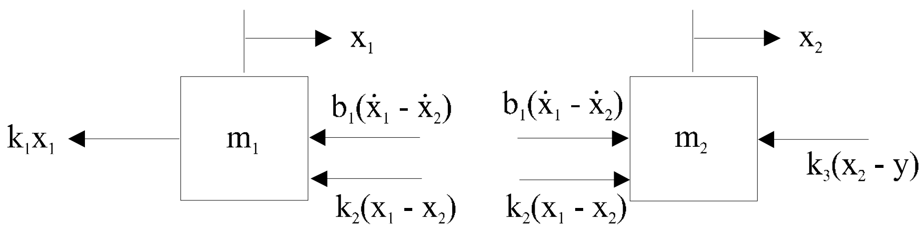
$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - b_1\dot{x}_1 - k_2\left(x_1 - x_2\right) \\
m_2\ddot{x}_2 &= -k_2\left(x_2 - x_1\right) - k_3\left(x_2 - y\right) - b_2\dot{x}_2
\end{aligned}
$$

Figure 2.38: Mechanical systems





$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1\dot{x}_1 \\
m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - k_3x_2
\end{aligned}
$$



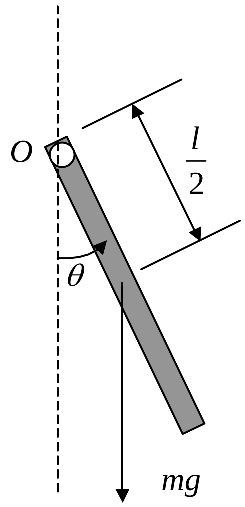
$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) \\
m_2\ddot{x}_2 &= F - k_2(x_2 - x_1) - b_1(\dot{x}_2 - \dot{x}_1)
\end{aligned}
$$

2. Write the equations of motion of a pendulum consisting of a thin, 2-kg stick of length l suspended from a pivot. How long should the rod be in order for the period to be exactly 2 secs? (The inertia I of a thin stick about an endpoint is $\frac{1}{3}ml^2$. Assume θ is small enough that $\sin\theta \cong \theta$.)

Solution:

Let's use Eq. (2.14)

$$M = I\alpha,$$



Moment about point O.

$$\begin{aligned} M_O &= -mg \times \frac{l}{2}\sin\theta = I_O\ddot{\theta} \\ &= \frac{1}{3}ml^2\ddot{\theta} \end{aligned}$$

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0$$

As we assumed θ is small,

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0$$

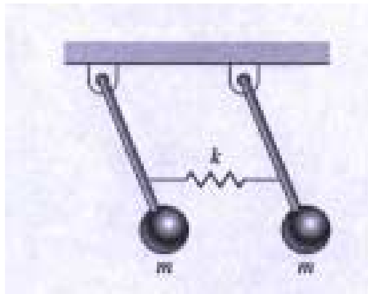
The frequency only depends on the length of the rod

$$\omega^2 = \frac{3g}{2l}$$

$$\begin{aligned} T &= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2l}{3g}} = 2 \\ l &= \frac{3g}{2\pi^2} = 1.49\,\mathrm{m} \end{aligned}$$

<Notes>

Figure 2.39: Double pendulum



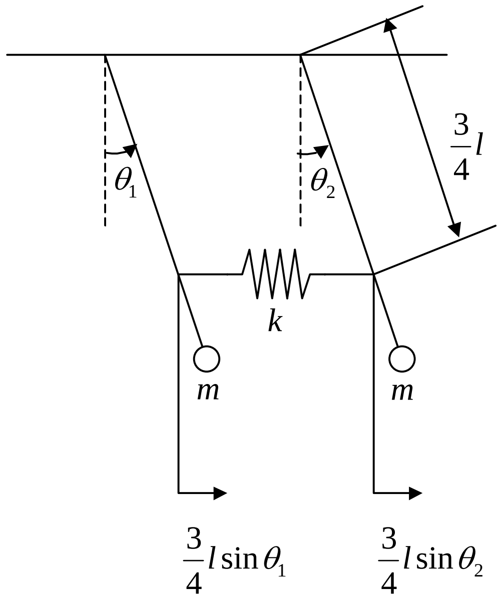
(a) Compare the formula for the period, $T = 2\pi\sqrt{\frac{2l}{3g}}$ with the well known formula for the period of a point mass hanging with a string with length l. $T = 2\pi\sqrt{\frac{l}{g}}$.

(b) Important!

In general, Eq. (2.14) is valid only when the reference point for the moment and the moment of inertia is the mass center of the body. However, we also can use the formular with a reference point other than mass center when the point of reference is fixed or not accelerating, as was the case here for point O.

3. Write the equations of motion for the double-pendulum system shown in Fig. 2.39. Assume the displacement angles of the pendulums are small enough to ensure that the spring is always horizontal. The pendulum rods are taken to be massless, of length l, and the springs are attached 3/4 of the way down.

Solution:



If we write the moment equilibrium about the pivot point of the left pendulem from the free body diagram,

$$M = -mgl\sin\theta_1 - k\frac{3}{4}l\left(\sin\theta_1 - \sin\theta_2\right)\cos\theta_1\frac{3}{4}l = ml^2\ddot{\theta}_1$$

$$ml^2\ddot{\theta}_1 + mgl\sin\theta_1 + \frac{9}{16}kl^2\cos\theta_1\left(\sin\theta_1 - \sin\theta_2\right) = 0$$

Similary we can write the equation of motion for the right pendulem

$$-mgl\sin\theta_2 + k\frac{3}{4}l\left(\sin\theta_1 - \sin\theta_2\right)\cos\theta_2\frac{3}{4}l = ml^2\ddot{\theta}_2$$

As we assumed the angles are small, we can approximate using $\sin\theta_1 \approx \theta_1, \sin\theta_2 \approx \theta_2$, $\cos\theta_1 \approx 1$, and $\cos\theta_2 \approx 1$. Finally the linearized equations of motion becomes,

$$\begin{aligned} ml\ddot{\theta}_1 + mg\theta_1 + \frac{9}{16}kl\left(\theta_1 - \theta_2\right) &= 0 \\ ml\ddot{\theta}_2 + mg\theta_2 + \frac{9}{16}kl\left(\theta_2 - \theta_1\right) &= 0 \end{aligned}$$

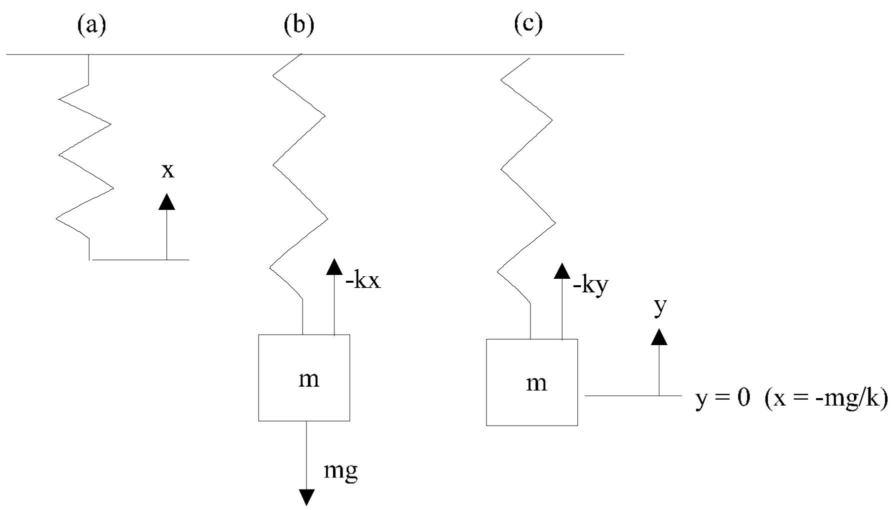
Or

$$\begin{aligned}\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{9}{16}\frac{k}{m}(\theta_1 - \theta_2) &= 0 \\ \ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{9}{16}\frac{k}{m}(\theta_2 - \theta_1) &= 0\end{aligned}$$

4. Write the equations of motion for a body of mass M suspended from a fixed point by a spring with a constant k. Carefully define where the body's displacement is zero.

Solution:

Some care needs to be taken when the spring is suspended vertically in the presence of the gravity. We define $x = 0$ to be when the spring is unstretched with no mass attached as in (a). The static situation in (b) results from a balance between the gravity force and the spring.



From the free body diagram in (b), the dynamic equation results

$$m\ddot{x} = -kx - mg.$$

We can manipulate the equation

$$m\ddot{x} = -k\left(x + \frac{m}{k}g\right),$$

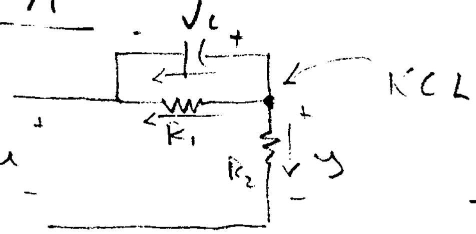
so if we replace x using $y = x + \frac{m}{k}g$,

$$\begin{aligned}m\ddot{y} &= -ky \\ m\ddot{y} + ky &= 0\end{aligned}$$

$$\frac{d^2\theta_1}{dt^2} = -\frac{g}{l}\theta_1 - \frac{9}{16}\frac{k}{m}(\theta_1-\theta_2) \qquad \frac{d^2\theta_2}{dt^2} = -\frac{g}{l}\theta_2 + \frac{9}{16}\frac{k}{m}(\theta_1-\theta_2)$$

Problem A

a)



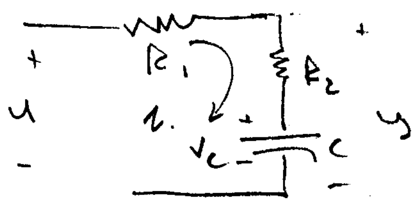
$$V_c = y - u$$

$$\frac{y}{R_2} + \frac{V_c}{R_1} + C\frac{dV_c}{dt} = 0$$

$$C\frac{dv_c}{dt} + \frac{v_c}{R_1} + \frac{v_c+u}{R_2} = 0$$

$$\Rightarrow C\frac{dv_c}{dt} + \left(\frac{1}{R_1}+\frac{1}{R_2}\right)v_c + \frac{u}{R_2} = 0$$

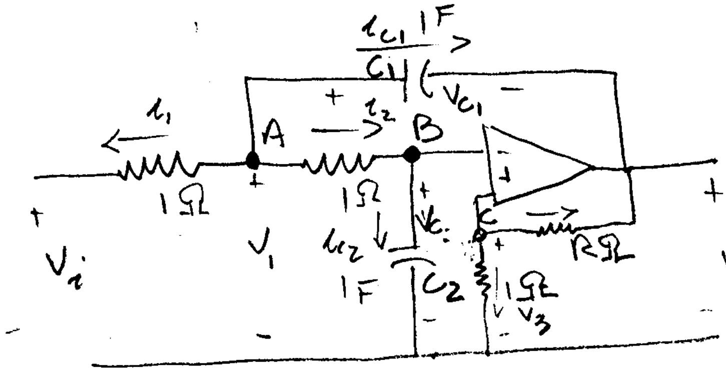
b)



$$R_1 i + R_2 i + V_c = u \qquad i = C\frac{dv_c}{dt}$$

$$C(R_1+R_2)\frac{dv_c}{dt} + v_c = u$$

c)



KCL at node A

$$\frac{V_1 - V_i}{1} + \frac{V_1 - V_{c_2}}{1} + i_{c_1} = 0$$

$$C_1 \frac{dV_{c_1}}{dt} = V_{c_2} - V_1 + V_i - V_1$$

$C_1 = 1$

(1) $$\frac{dV_{c_1}}{dt} = V_{c_2} - 2V_1 + V_i$$

KCL at node B

$$i_2 = i_{c_2} \rightsquigarrow C_2 \frac{dV_{c_2}}{dt} = \frac{V_1 - V_{c_2}}{1}$$

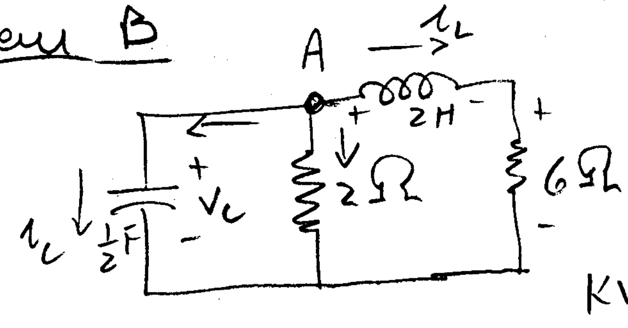
$C_2 = 1 \rightsquigarrow$ (2) $$\frac{dV_{c_2}}{dt} = V_1 - V_{c_2}$$

KCL at node C

(3) $$\frac{V_3}{1} + \frac{V_3 - V_o}{R} = 0$$, ideal op amps $\rightsquigarrow V_{c_2} = V_3$

and (4) $V_{c_1} + V_o = V_1$

Problem B



KCL at node A

$$i_c + \frac{V_c}{2} + i_L = 0$$

$$i_c = \frac{1}{2} \frac{dV_c}{dt} \rightsquigarrow \frac{dv_c}{dt} = -2i_L - V_c$$

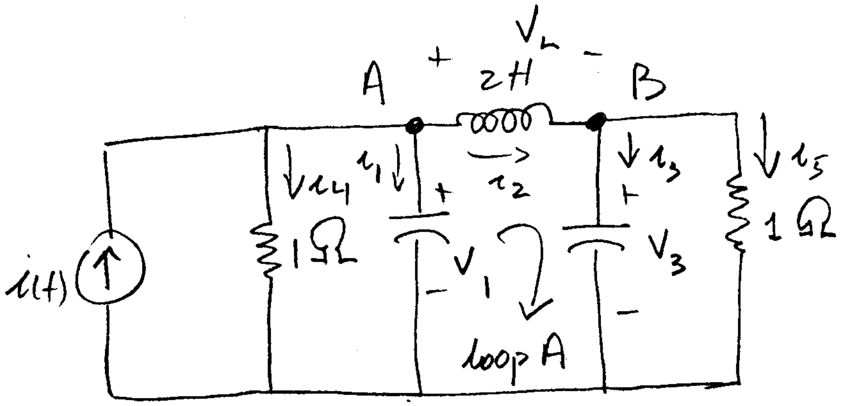
KVL around outside loop

$$2\frac{di_L}{dt} + 6i_L = V_c \rightsquigarrow \frac{di_L}{dt} = \frac{1}{2}V_c - 3i_L$$

$x_1 = V_c \quad x_2 = i_L \implies$

$$\begin{bmatrix} \frac{dv_c}{dt} \\ \frac{di_L}{dt} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ \frac{1}{2} & -3 \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix}$$

no exter[nal] inp.



KCL at A

$$i = i_4 + i_1 + i_2$$

KCL at B

$$i_2 = i_3 + i_5$$

$$i_2 = \frac{dV_3}{dt} + V_3$$

$$i_1 = \frac{dV_1}{dt}, \quad i_3 = \frac{dV_3}{dt} \qquad i_5 = \frac{V_3}{1} = V_3$$

state variables are V_1, V_3, i_2

KVL around loop A

$$V_1 = V_3 + 2\frac{di_2}{dt}$$

$$i(t) = V_1 + \frac{dV_1}{dt} + i_2 \rightsquigarrow \frac{dV_1}{dt} = i(t) - V_1 - i_2$$

$$\begin{bmatrix} \frac{dV_1}{dt} \\ \frac{di_2}{dt} \\ \frac{dV_3}{dt} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ i_2 \\ V_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} i(t)$$