1. (a) Show that α_0 in Eq. (6.2) is given by

$$\alpha_0 = \left[G(s)\frac{U_0\omega}{s-j\omega}\right]_{s=-j\omega} = -U_0G(-j\omega)\frac{1}{2j}$$

and

$$\alpha_0^* = \left[G(s)\frac{U_0\omega}{s+j\omega}\right]_{s=+j\omega} = U_0G(j\omega)\frac{1}{2j}.$$

(b) By assuming the output can be written as

$$y(t) = \alpha_0 e^{-j\omega t} + \alpha_0^* e^{j\omega t},$$

derive Eqs. (6.4) - (6.6).

Solution:

(a) Eq. (6.2):

$$Y(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots + \frac{\alpha_n}{s-p_n} + \frac{\alpha_o}{s+j\omega_o} + \frac{\alpha_o^*}{s-j\omega_o}$$

Multiplying this by $(s+jw)$:

$$\Rightarrow \alpha_0 = Y(s)(s+jw) - \frac{\alpha_1}{s-p_1}(s+jw) - \ldots - \frac{\alpha_n}{s-p_n}(s+jw) - \frac{\alpha_0^*}{s-jw}(s+jw)$$

$$\alpha_0 = \alpha_0\,|_{s=-jw} = [Y(s)(s+jw) - \frac{\alpha_1}{s-p_1}(s+jw) - \ldots - \frac{\alpha_0^*}{s-jw}(s+jw)]_{s=-jw}$$

$$= Y(s)(s+jw)\,|_{s=-jw} = G(s)\frac{U_0w}{s^2+w^2}(s+jw)\,|_{s=-jw}$$

$$= G(s)\frac{U_0w}{s-jw}\,|_{s=-jw} = -U_0G(-jw)\frac{1}{2j}$$

Similarly, multiplying Eq. (6.2) by $(s-jw)$:

$$Y(s)(s-jw) = \frac{\alpha_1}{s-p_1}(s-jw) + \ldots + \frac{\alpha_n}{s-p_n}(s-jw) + \frac{\alpha_0}{s+jw}(s-jw) + \alpha_0^*$$

$$\alpha_0^* = \alpha_0^*\,|_{s=jw} = Y(s)(s-jw)\,|_{s=jw} = G(s)\frac{U_0w}{s^2+w^2}(s-jw)\,|_{s=jw}$$

$$= G(s)\frac{U_0w}{s+jw}\,|_{s=jw} = U_0G(jw)\frac{1}{2j}$$

(b)

$$\begin{aligned}
y(t) &= \alpha_o e^{-j\omega t} + \alpha_o^* e^{j\omega t} \\
y(t) &= -U_o G(-j\omega)\frac{1}{2j}e^{-j\omega t} + U_o G(j\omega)\frac{1}{2j}e^{j\omega t} \\
&= U_o\left[\frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j}\right] \\
|G(j\omega)| &= \left\{\mathrm{Re}\,[G(j\omega)]^2 + \mathrm{Im}\,[G(j\omega)]^2\right\}^{\frac{1}{2}} = A \\
\angle G(j\omega) &= \tan^{-1}\frac{\mathrm{Im}\,[G(j\omega)]}{\mathrm{Re}\,[G(j\omega)]} = \phi \\
|G(-j\omega)| &= \left\{\mathrm{Re}\,[G(-j\omega)]^2 + \mathrm{Im}\,[G(-j\omega)]^2\right\}^{\frac{1}{2}} = |G(j\omega)| \\
&= \left\{\mathrm{Re}\,[G(j\omega)]^2 + \mathrm{Im}\,[G(j\omega)]^2\right\}^{\frac{1}{2}} = A \\
\angle G(-j\omega) &= \tan^{-1}\frac{\mathrm{Im}\,[G(-j\omega)]}{\mathrm{Re}\,[G(-j\omega)]} = \tan^{-1}\frac{-\,\mathrm{Im}\,[G(j\omega)]}{\mathrm{Re}\,[G(j\omega)]} = -\phi \\
&\Rightarrow\ G(j\omega) = Ae^{j\phi},\ G(-j\omega) = Ae^{-j\phi}
\end{aligned}$$

Thus,

$$\begin{aligned}
y(t) &= U_o\left[\frac{Ae^{j\phi}e^{j\omega t} - Ae^{-j\phi}e^{-j\omega t}}{2j}\right] = U_o A\left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right] \\
y(t) &= U_o A\sin(\omega t + \phi)
\end{aligned}$$

where

$$A = |G(j\omega)|\,,\ \phi = \tan^{-1}\frac{\mathrm{Im}\,[G(j\omega)]}{\mathrm{Re}\,[G(j\omega)]} = \angle G(j\omega)$$

2. (a) Calculate the magnitude and phase of

$$G(s) = \frac{1}{s+10}$$

by hand for $\omega = 1$, 2, 5, 10, 20, 50, and 100 rad/sec.

(b) sketch the asymptotes for $G(s)$ according to the Bode plot rules, and compare these with your computed results from part (a).

Solution:

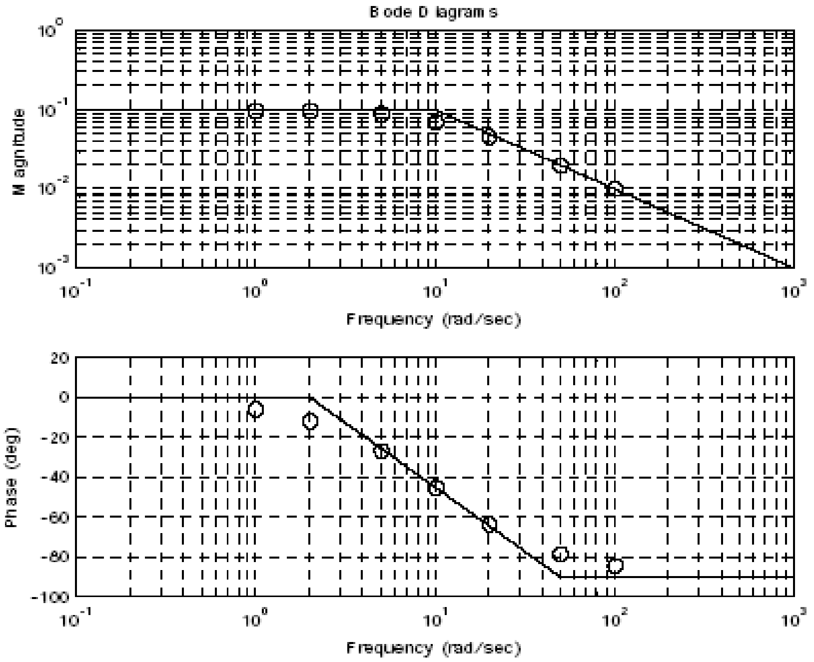
(a)

$$G(s) = \frac{1}{s+10},\ G(j\omega) = \frac{1}{10+j\omega} = \frac{10-j\omega}{100+\omega^2}$$

$$|G(j\omega)| = \frac{1}{\sqrt{100+\omega^2}},\ \angle G(j\omega) = -\tan^{-1}\frac{\omega}{10}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
1	0.0995	−5.71
2	0.0981	−11.3
5	0.0894	−26.6
10	0.0707	−45.0
20	0.0447	−63.4
50	0.0196	−78.7
100	0.00995	−84.3

(b) The Bode plot is :



6.3 (b)

$$L(s) = \frac{100}{s(0.1s+1)(0.5s+1)}$$

Solution:

Step 1. Convert the function to Bode form of Eq. (6.16) in text.

$$L(j\omega) = \frac{100}{j\omega(j\omega/10+1)(j\omega/2+1)}$$

Step 2. We note the $j\omega$ term is first-order and in the denominator, so n = -1. Therefore the low-frequency asymptote is defined by the first term:

$$L(j\omega) = \frac{100}{j\omega}$$

The magnitude plot of this term has the slope of -1 (or -20db per decade). We locate the magnitude by passing through the value 2 at $\omega = 1$.

Step 3. The first break point is at $\omega = 2$ and is a first order term in the denominator. We draw a -2 slope line that intersects the previous -1 slope at $\omega = 2$. Finally, we draw a -3 slope line that intersects the previous -2 slope at $\omega = 10$.

Step 4. We sketch in the actual curve so that it is approximately tangent to the asymptotes when far away from the break points, a factor 0f 0.7(-3db) below the asymptote at the $\omega = 2$ and $\omega = 10$ break points.

Step 5. Because the phase of $\frac{100}{j\omega}$ is -90^o, the phase curve starts at -90^o at the lowest frequency.

Step 6. As a guide, the phase changes to -180^o at $\omega = 2$ and -270^o at $\omega = 10$.

Step 7. Locate the asymptotes for each individual phase curve so that their phase change corresponds to the steps in the phase toward or away from the approximate curve indicated by Step 6.

Step 8. Graphically add each phase curve.

Refer figure [1].

6.3 (i)

$$L(s) = \frac{s}{(s+1)(s+10)(s^2+2s+2500)}$$

Solution:

This system contains a second-order term in denominator. We convert the transfer function to the Bode form of Eq.(6.16) in text.

$$L(s) = \frac{\frac{1}{25000}s}{(s+1)(\frac{s}{10}+1)[(\frac{s}{50})^2 + 2\frac{1}{50}\frac{s}{50}+1]}$$

Starting with the low-frequency asymptote, we have n=1 and $|L(j\omega)| \approx \frac{1}{25000}\omega$. The magnitude plot of this term has a slope of +1(20db per decade). $\omega = 1$ and $\omega = 10$ are break-points. The slope shifts to 0 and -1 at $\omega = 1$ and $\omega = 10$, respectively. We locate the magnitude by passing through the value $\frac{1}{25000}$ at $\omega = 1$ even though the composite curve will not go through this point because the break point at $\omega = 1$. For the second-order pole, note that $\omega_n = 50$ and

$\xi = 0.02$. At the break-point frequency of the poles, $\omega = 50$, the slope shifts to -2(-40db per decade). At the pole break point the magnitude ratio above the asymptote is $\frac{1}{2\xi} = 25$.

The phase curve for this case starts at $\phi = 90^o$ corresponding to s term, falls to $\phi = 0^o$ at $\omega = 1$, and then approaches to $\phi = -90^o$ at $\omega = 10$ and finally approaches to $\phi = -270^o$.

Refer figure [2].

6.4 (b)

$$L(s) = \frac{(s+2)}{s(s+1)(s+5)(s+10)}$$

Solution:

Step 1. Convert the function to Bode form of Eq. (6.16) in text.

$$L(j\omega) = \frac{\frac{1}{25}(\frac{j\omega}{2}+1)}{j\omega(j\omega+1)(j\omega/5+1)(j\omega/10+1)}$$

Step 2. We note the $j\omega$ term is first-order and in the denominator, so n = -1. Therefore the low-frequency asymptote is defined by the first term:

$$L(j\omega) = \frac{1}{25j\omega}$$

The magnitude plot of this term has the slope of -1 (or -20db per decade). We locate the magnitude by passing through the value 0.04 at $\omega = 1$.

Step 3. The first break point is at $\omega = 1$ and is a first order term in the denominator. We draw a -2 slope line that intersects the previous -1 slope at $\omega = 1$. Break point at $\omega = 2$ is a first term in the numerator, thus calls for a change in slope of +1. Then we draw a line with -1 slope. At breakpoints at $\omega = 5$ and $\omega = 10$ the slope changes to -2 and -3, respectively.

Step 4. We sketch in the actual curve so that it is approximately tangent to the asymptotes when far away from the break points, a factor 0f 0.7(-3db) below the asymptote at the $\omega = 1$, $\omega = 5$ and $\omega = 10$ break points, while a factor of 1.4 beyond the asymptote at the $\omega = 2$.

Step 5. Because the phase of $\frac{25}{j\omega}$ is -90^o, the phase curve starts at -90^o at the lowest frequency.

Step 6. As a guide, the phase changes to -180^o at $\omega = 1$, -90^o at $\omega = 2$, -180^o at $\omega = 5$ and finally -270^o at $\omega = 10$.

Step 7. Locate the asymptotes for each individual phase curve so that their phase change corresponds to the steps in the phase toward or away from the approximate curve indicated by Step 6.

Step 8. Graphically add each phase curve.

Refer figure [3].

6.5 (b)

$$L(s) = \frac{1}{s(s^2+3s+10)}$$

Solution:

This system contains a second-order term in denominator. We convert the transfer function to the Bode form of Eq.(6.16) in text.

$$L(s) = \frac{\frac{1}{10}}{s[(\frac{s}{\sqrt{10}})^2 + 2\frac{3s}{20} + 1]}$$

Starting with the low-frequency asymptote, we have n=-1 and $|L(j\omega)| \approx \frac{1}{10\omega}$. The magnitude plot of this term has a slope of -1(-20db per decade). For the second-order pole, note that $\omega_n = \sqrt{10}$ and $\xi = 0.4743$. At the break-point frequency of the poles, $\omega = \sqrt{10}$, the slope shifts to -3(-60db per decade). At the pole break point the magnitude ratio above the asymptote is $\frac{1}{2\xi} = 1.05$.

The phase curve for this case starts at $\phi = -90^o$, falls to $\phi = -270^o$ at $\omega = \sqrt{10}$.

Refer figure [4].

6.6 (f)

$$L(s) = \frac{(s+1)^2}{s^3(s+4)}$$

Solution:

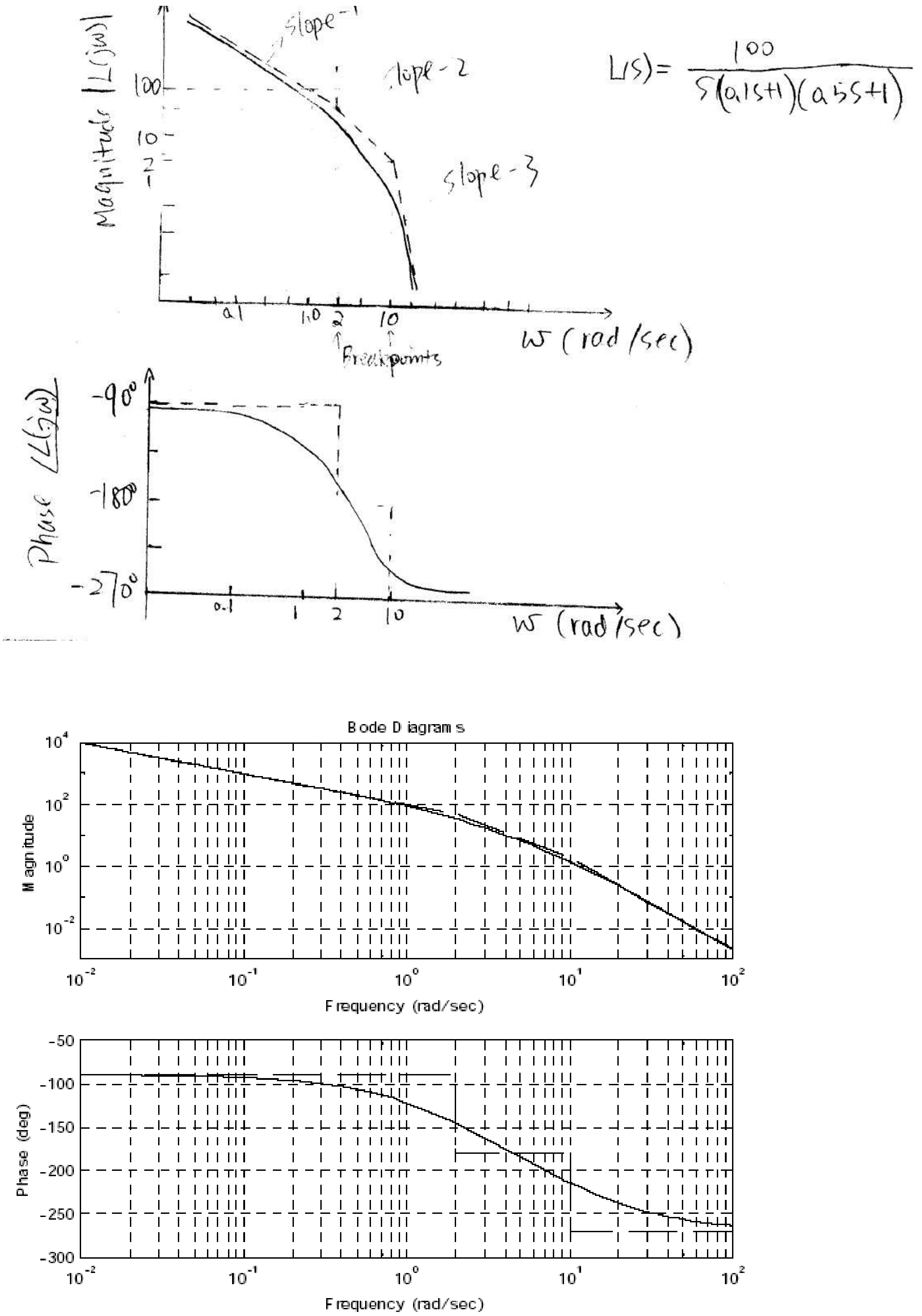
We convert the transfer function to the Bode form.

$$L(s) = \frac{\frac{1}{4}(s+1)^2}{s^3(\frac{s}{4}+1)}$$

Starting with the low-frequency asymptote, we have n=-3 and $|L(j\omega)| \approx \frac{1}{4\omega^3}$. The magnitude plot of this term has a slope of -3. At the breakpoint $\omega = 1$, the slope changes to -1. Finally, at the pole breakpoint $\omega = 4$, the slope shifts to -2.

The phase curve for this case starts at $\phi = -270^o$. It changes to -90^o at $\omega = 1$, falls to $\phi = -180^o$ at $\omega = 4$.

Refer figure [5].



$$L(s) = \frac{100}{s(0.1s+1)(0.5s+1)}$$

Figure 1: solution of 6.3(b)

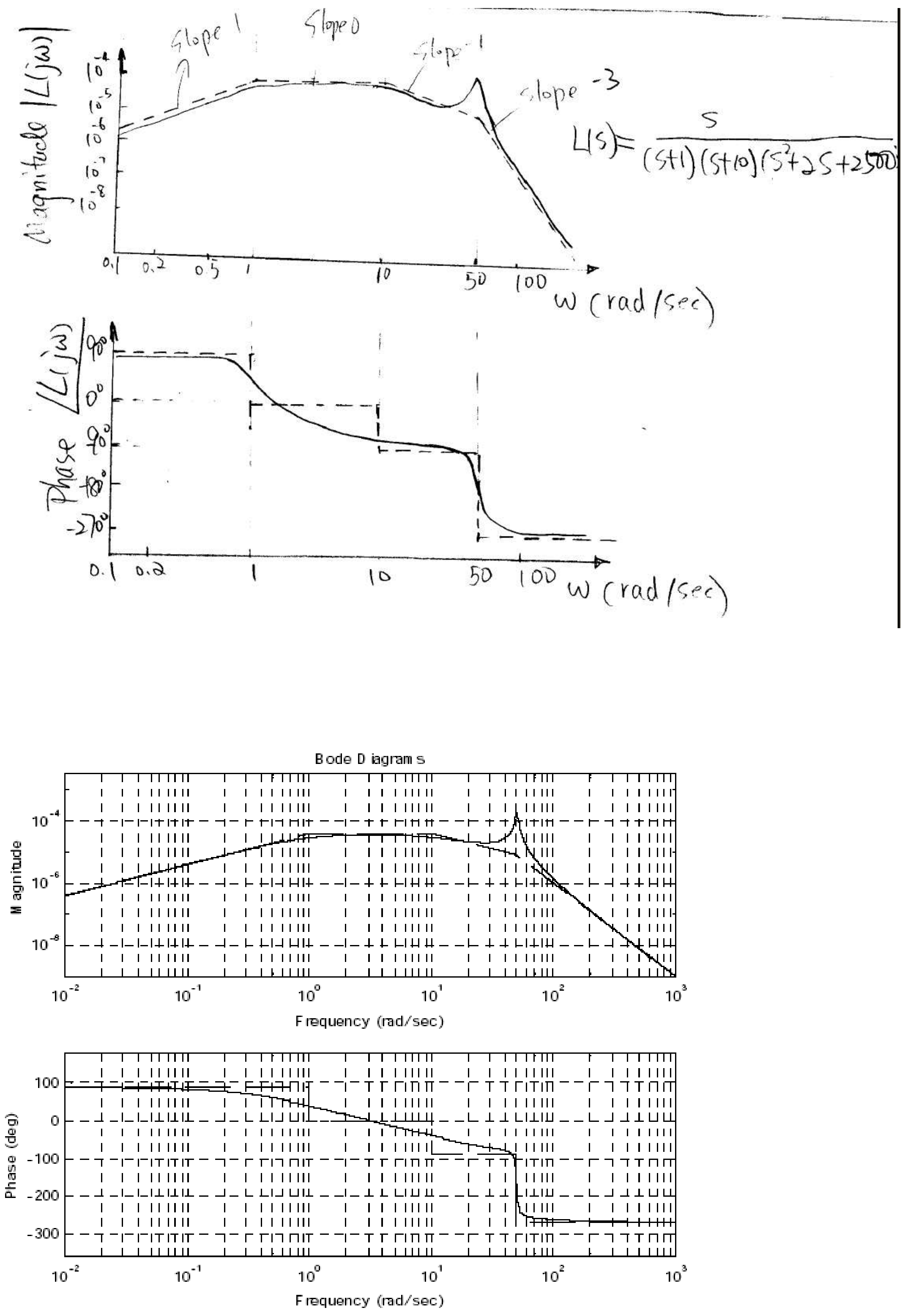
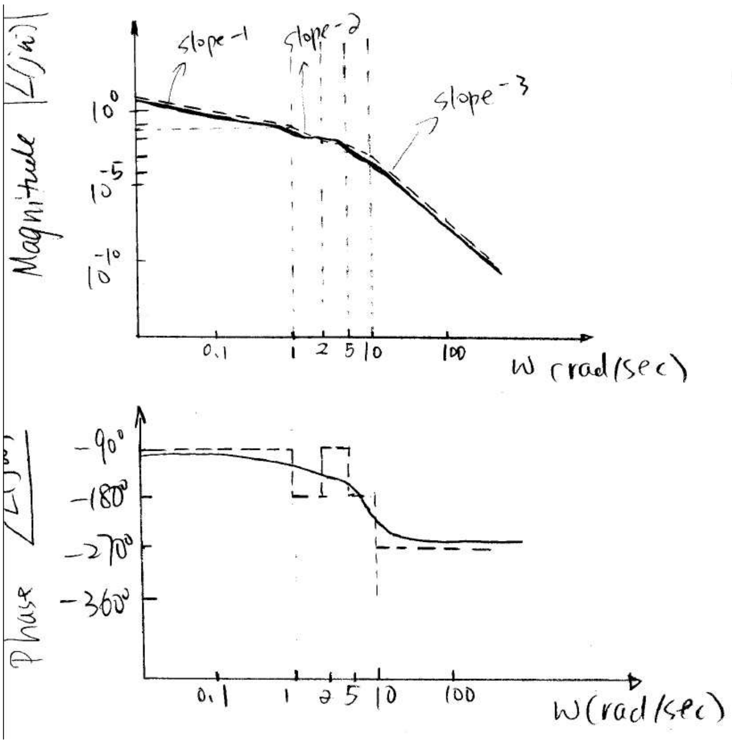


Figure 2: solution of 6.3(i)



$$L(s) = \frac{(s+2)}{s(s+1)(s+5)(s+10)}$$

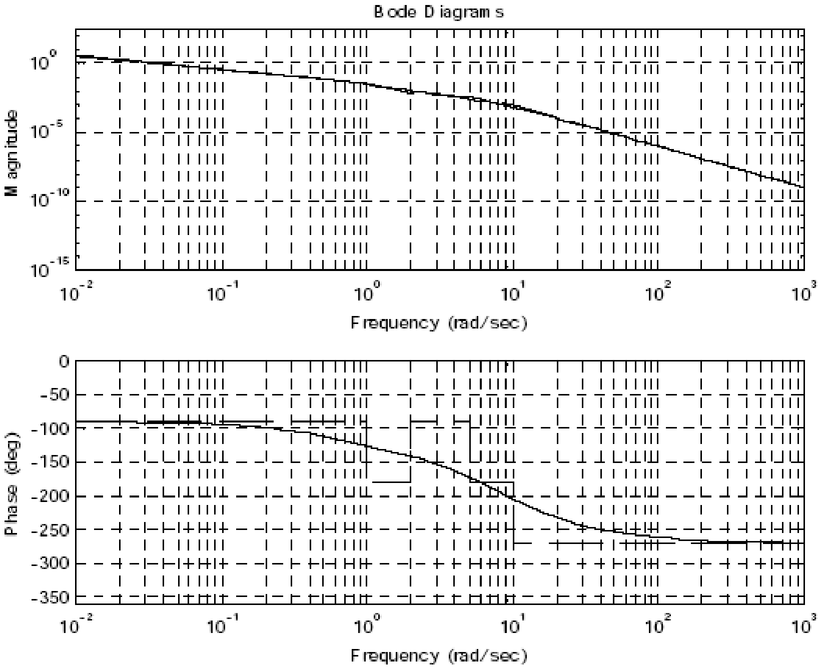


Figure 3: solution of 6.4(b)

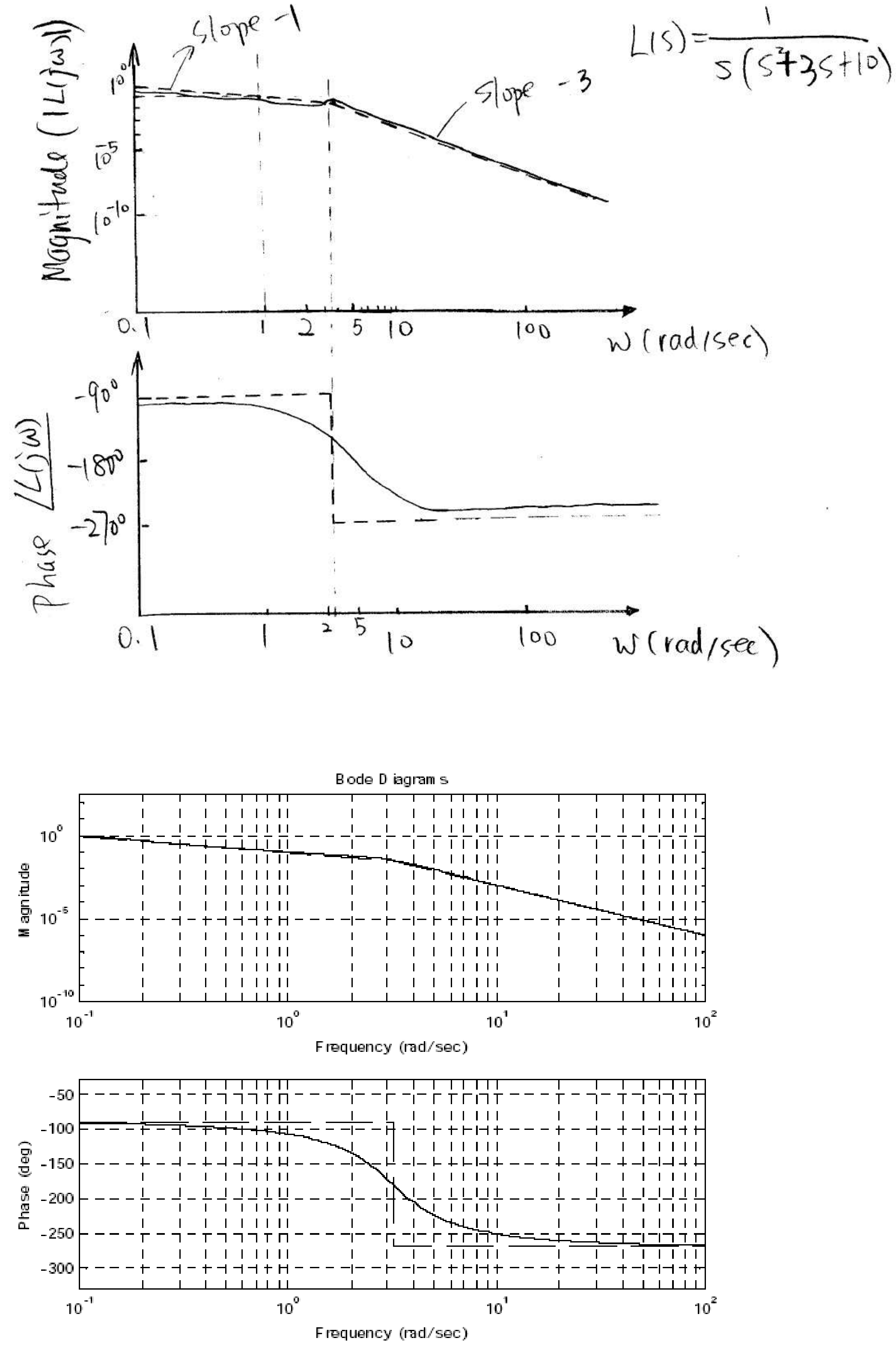
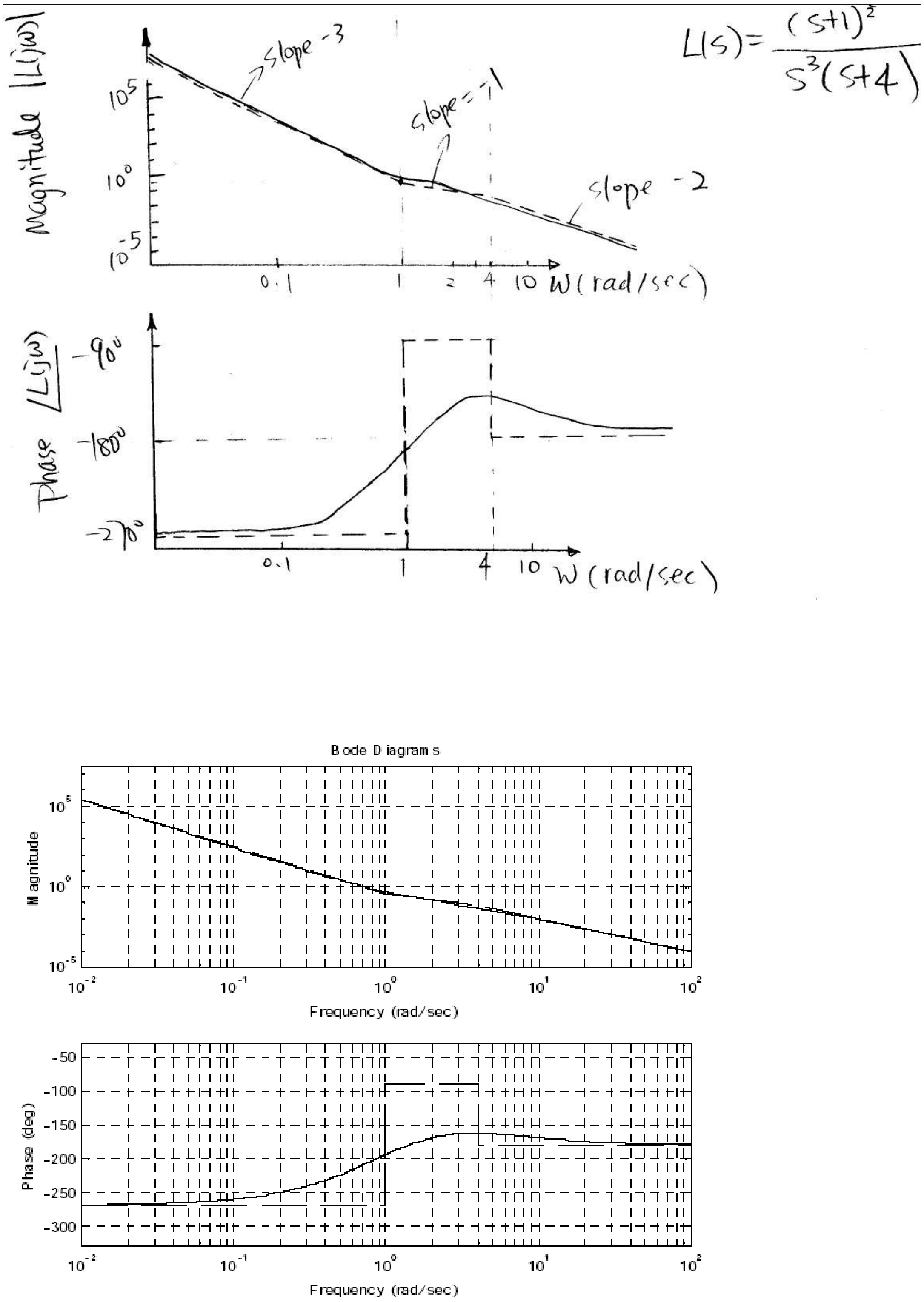


Figure 4: solution of 6.5(b)



$$L(s) = \frac{(s+1)^2}{s^3(s+4)}$$

Figure 5: solution of 6.6(f)